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ON QUADRATIC INTEGRALS OF LINEAR AUTONOMOUS SYSTEMS

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Quadratic integrals of the systems of linear differential equations with constant coefficients were studied in [1 – 3]. Below a method of using a known quadratic integral to construct other quadratic integrals of such systems is given.

Let us consider the system

$$x^{\cdot} = Ax \tag{1}$$

where A is a $n \times n$ matrix. We assume that the quadratic integral

$$V = x'Bx, \quad B = B', \quad A'B + BA = 0 \tag{2}$$

has been found for the system (1). The following question now arises: knowing the integral (1), it is possible to find one or more independent quadratic integrals of the system (1).

Theorem 1. If the system (1) has the quadratic integral (2), then it also has the following quadratic integrals:

$$W = x'Cx, \quad C = C' = L'BM + M'BL \tag{3}$$

where L and M are arbitrary constant $n \times n$ matrices commuting with the matrix A. If in addition the determinant $|B| \neq 0$, then all quadratic integrals of the system (1) are contained in the formula (3).

Proof. Direct calculation of the total time derivative of the quadratic form W shows, by virtue of the system (1), that the form is an integral of the system (1). We shall prove that when $|B| \neq 0$ then any quadratic integral

$$U = x'Dx, \quad D = D' \tag{4}$$

of (1) can be written in the form (3). Indeed, setting in (3) $L = {}^1/_2\, E$ (E is the unit matrix) and $M = B^{-1}D$, we obtain

$$W = x' \, [{}^1/_2\, E'BB^{-1}D + {}^1/_2\, D' \, (B^{-1})'BE]x = x'Dx = U$$

The matrix E commutes with A, and the commutativity of the matrices $B^{-1}D$ and A is proved in the following manner. From (2) it follows that $A' = -BAB^{-1}$. Substituting this into the equation $A'D + DA = 0$ we obtain $BAB^{-1}D + DA = 0$ or $AB^{-1}D = B^{-1}DA$, QED. Consequently when $|B| \neq 0$, all quadratic integrals of the system (1) are contained within the formula (3) and this completes the proof of the theorem.

Corollary 1. If the system (1) has the quadratic integral (2), then it also has the following quadratic integrals:

$$Q = x' \, [(P')^k BP^l + (P')^l BP^k] \, x, \qquad k, \, l = 1, \ldots, \, n - 1 \tag{5}$$

where P represents an arbitrary constant $n \times n$ matrix commuting with A (e.g. $P = A$).

Note 1. If in (5) $k, l \geqslant n$, then the integrals obtained will be linearly dependent on the integrals (5).

Let us now consider the problem of independence of the quadratic integrals of linear autonomous systems. Applying the known theorem on the independence of functions to the present case, we obtain

Theorem 2. The quadratic integrals

$$V_i = x'B_ix, \; B_i = B_i', \quad i = 1, \ldots, m$$

of the system (1) are independent if and only if the relation

$$\operatorname{rank}\,(B_1x_0, B_2x_0, \ldots, B_mx_0) = m \tag{6}$$

holds for some $x = x_0$.

Corollary 2. If the system (1) has the quadratic integral (2), then this integral together with the integrals obtained from it by means of the formula (3)

$$V_i = x'B_ix, \; B_i = L_i'BM_i + M_i'BL_i, \quad i = 1, \ldots, m-1$$

will be independent if and only if condition (6) in which $B_m = B_0$ holds.
When $m = 2$, Theorem 2 can be formulated as follows.

Theorem 3. Two quadratic integrals of the system (1) are independent if and only if they are linearly independent.

Proof. Clearly it is sufficient to prove that the linear dependence of the integrals (2) and (4) implies the linear dependence of the matrices B and D. The condition of dependence of two integrals on each other has, in the case of the integrals (2) and (4), has the form $\operatorname{rank}\,(Bx, Dx) < 2$. It can be shown that the latter relation holds if and only if $B = \lambda D$ where $\lambda = \text{const}$, and this proves the theorem.

Corollary 3. If the matrices L and M of Theorem 1 satisfy the condition $L'BM + M'BL \neq \lambda B$, the integrals (2) and (3) are independent.

Thus it follows, that to establish the independence of two quadratic integrals of the system (1) on each other, it is sufficient to show that they are linearly independent. If on the other hand we have more than two quadratic integrals of the system (1) then, generally speaking, their linear independence is insufficient for their independence. Indeed, consider the system (1) with quadratic integrals $V_1 = x_1^2$, $V_2 = x_2^2$, $V_3 = x_1x_2$. They are obviously independent, but at the same time they are connected by the relation $V_3^2 = V_1V_2$, i.e. they are dependent on each other.

Example. We know that the system

$$y^{\cdot\cdot} + Gy^{\cdot} + Fy = 0 \tag{7}$$

where $G = -G'$ and $F = F'$ are constant $n \times n$ matrices, has a quadratic integral (the energy integral)

$$H = {}^1/_2\,(y^{\cdot\prime}Ey^{\cdot} + y'Fy) \tag{8}$$

Another quadratic integral of the system of the form (7) with $n = 2$

$$y_1^{\cdot\cdot} - py_2^{\cdot} - ay_1 = 0, \quad y_2^{\cdot\cdot} + py_1^{\cdot} - by_2 = 0$$

was given in [4 — 6]

$$H_1 = 2(by_1^{\cdot}y_2 - ay_1y_2^{\cdot}) - p(ay_1^2 + by_2^2) + \frac{b-a}{2p}(y_2^{\cdot 2} - y_1^{\cdot 2} + ay_1^2 - by_2^2)$$

Let us write the system (7) in the form (1) and the integral (8) in the form (2), setting $x' = (y', y^{\cdot\prime})$. We obtain

$$A = \begin{Vmatrix} 0 & E \\ -F & -G \end{Vmatrix}, \quad B = \begin{Vmatrix} {}^1/_2 F & 0 \\ 0 & {}^1/_2 E \end{Vmatrix}$$

The formula (3) yields, in certain cases, new quadratic integrals of the system (7). For example, the following quadratic integral is given in [7] for the system (7):

$$H_2 = (Gy^{\cdot} + Fy)' \, (Gy^{\cdot} + Fy) + y^{\cdot\prime} Fy^{\cdot} \tag{9}$$

This is obtained from (8) using the formula (5) and setting $P = A$, $k = l = 1$. If the determinant $|F| \neq 0$, then according to Theorem 1 all quadratic integrals of the system (8) are given by the formula (3).

Note 2. It can be confirmed that the integral H_1 can be expressed linearly in terms of the integrals (8) and (9) $(n = 2)$

$$H_1 = -\frac{a+b-2p^2}{p} H - \frac{1}{p} H_2$$

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